

On the Deligne-Simpson problem and its weak version

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Abstract

We consider the *Deligne-Simpson problem (DSP)* (resp. the weak DSP): Give necessary and sufficient conditions upon the choice of the $p+1$ conjugacy classes $c_j \subset \mathfrak{gl}(n, \mathbf{C})$ or $C_j \subset GL(n, \mathbf{C})$ so that there exist irreducible $(p+1)$ -tuples (resp. $(p+1)$ -tuples with trivial centralizers) of matrices $A_j \in c_j$ with zero sum or of matrices $M_j \in C_j$ whose product is I . The matrices A_j (resp. M_j) are interpreted as matrices-residua of Fuchsian linear systems (resp. as monodromy matrices of regular linear systems) of differential equations with complex time. In the paper we give sufficient conditions for solvability of the DSP in the case when one of the matrices is with distinct eigenvalues.

1 Introduction

1.1 Basic notions and purpose of this paper

In the present paper we consider the *Deligne-Simpson problem (DSP)*: Give necessary and sufficient conditions upon the choice of the $p+1$ conjugacy classes $c_j \subset \mathfrak{gl}(n, \mathbf{C})$ or $C_j \subset GL(n, \mathbf{C})$ so that there exist irreducible $(p+1)$ -tuples of matrices $A_j \in c_j$ satisfying the condition

$$A_1 + \dots + A_{p+1} = 0 \quad (1)$$

or of matrices $M_j \in C_j$ satisfying the condition

$$M_1 \dots M_{p+1} = I \quad (2)$$

Convention 1 In what follows we write “tuple” instead of “ $(p+1)$ -tuple” and the matrices A_j (resp. M_j) are always supposed to satisfy condition (1) (resp. (2)).

The matrices A_j (resp. M_j) are interpreted as matrices-residua of a Fuchsian system of linear differential equations (resp. as monodromy matrices of a regular linear system) on Riemann's sphere; see a more detailed description in [Ko1] or [Ko2].

Remark 2 The version with matrices A_j (resp. M_j) is called the additive (resp. the multiplicative) version of the DSP. The multiplicative version of the problem was formulated by P. Deligne and C. Simpson was the first to obtain results towards its resolution, see [Si1] and [Si2]. The additive version is due to the author.

We presume the necessary condition $\prod \det(C_j) = 1$ (resp. $\sum \text{Tr}(c_j) = 0$) to hold. In terms of the eigenvalues $\sigma_{k,j}$ (resp. $\lambda_{k,j}$) of the matrices from C_j (resp. c_j) repeated with their multiplicities, this condition reads $\prod_{k=1}^n \prod_{j=1}^{p+1} \sigma_{k,j} = 1$ (resp. $\sum_{k=1}^n \sum_{j=1}^{p+1} \lambda_{k,j} = 0$).

Definition 3 An equality $\prod_{j=1}^{p+1} \prod_{k \in \Phi_j} \sigma_{k,j} = 1$, resp. $\sum_{j=1}^{p+1} \sum_{k \in \Phi_j} \lambda_{k,j} = 0$, is called a non-genericity relation; the sets Φ_j contain one and the same number $N < n$ of indices for all j (when wishing to specify N we say “ N -relation” instead of “non-genericity relation”). Eigenvalues satisfying none of these relations are called generic.

Remarks 4 1) Reducible tuples of matrices A_j or M_j exist only for non-generic eigenvalues (the eigenvalues of each diagonal block of a block upper-triangular tuple satisfy some non-genericity relation). Therefore for generic eigenvalues existence of tuples implies automatically their irreducibility. This is not true for non-generic eigenvalues.

2) It is clear that the presence of a non-genericity relation with $N = N_0$ implies the presence of one with $N = n - N_0$ (just replace the sets Φ_j by their complements in $\{1, 2, \dots, n\}$). Therefore in what follows we consider only non-genericity relations with $N \leq n/2$.

Part 1) of the above remarks explains why for non-generic eigenvalues it is reasonable to require instead of irreducibility of the tuple only triviality of its centralizer (i.e. only scalar matrices to commute with all matrices from the tuple). This is the *weak version of the DSP* (or just the *weak DSP* for short).

Definition 5 We say that the DSP (resp. the weak DSP) is solvable for a given tuple of conjugacy classes c_j or C_j if there exist irreducible tuples of matrices $A_j \in c_j$ or $M_j \in C_j$ (resp. if there exist tuples of such matrices with trivial centralizers).

We assume throughout the paper that there holds

Convention 6 The conjugacy classes c_1 and C_1 are with distinct eigenvalues.

The purpose of the present paper is to show as precisely as possible where passes the border between the cases when the DSP is solvable and when it is not but the weak DSP is solvable.

1.2 The known results

Definition 7 Call Jordan normal form (JNF) of size n a family $J^n = \{b_{i,l}\}$ ($i \in I_l$, $I_l = \{1, \dots, s_l\}$, $l \in L$) of positive integers $b_{i,l}$ whose sum is n . Here L is the set of indices of eigenvalues (all distinct) and I_l is the set of indices of Jordan blocks with eigenvalue l , $b_{i,l}$ is the size of the i -th block with this eigenvalue. An $n \times n$ -matrix Y has the JNF J^n (notation: $J(Y) = J^n$) if to its distinct eigenvalues λ_l , $l \in L$, there belong Jordan blocks of sizes $b_{i,l}$. We use the following notation (illustrated by an example): the JNF $\{\{3, 2\}, \{7, 6, 1\}\}$ is the one with two eigenvalues to the first (to the second) of which there belong two blocks, of sizes 3 and 2 (resp. three blocks, of sizes 7, 6 and 1).

Notation 8 1) We denote by $C(Y)$ the conjugacy class (in $gl(n, \mathbf{C})$ or $GL(n, \mathbf{C})$) of the matrix Y . We set $C(Y) = C(X) \times C(Z)$ if $Y = \begin{pmatrix} X & 0 \\ 0 & Z \end{pmatrix}$ (here X is $l \times l$ and Z is $(n-l) \times (n-l)$).

2) For a conjugacy class C in $GL(n, \mathbf{C})$ or $gl(n, \mathbf{C})$ denote by $d(C)$ its dimension and by $J(C)$ the JNF it defines. For a matrix $Y \in C$ set $r(C) := \min_{\lambda \in \mathbf{C}} \text{rank}(Y - \lambda I)$. The integer $n - r(C)$ is the maximal number of Jordan blocks of $J(Y)$ with one and the same eigenvalue. Set $d_j := d(C_j)$ (resp. $d(c_j)$), $r_j := r(C_j)$ (resp. $r(c_j)$). The quantities $r(C)$ and $d(C)$ depend only on the JNF $J(Y) = J^n$, not on the eigenvalues, so we write sometimes $r(J^n)$ and $d(J^n)$.

Proposition 9 (*C. Simpson, see [Si1].*) *The following couple of inequalities is a necessary condition for the existence of irreducible $(p+1)$ -tuples satisfying (2) or (1):*

$$d_1 + \dots + d_{p+1} \geq 2n^2 - 2 \quad (\alpha_n)$$

$$\text{for all } j, r_1 + \dots + \hat{r}_j + \dots + r_{p+1} \geq n \quad (\beta_n)$$

The above proposition holds without Convention 6. When Convention 6 holds, then $r_1 = n - 1$ and condition (β_n) is tantamount to $r_2 + \dots + r_{p+1} \geq n$.

Definition 10 *The quantity $\kappa = 2n^2 - d_1 - \dots - d_{p+1}$ (see Notation 8) is called the index of rigidity of a given tuple of conjugacy classes or of JNFs. It has been introduced by N.Katz, see [Ka]. If condition (α_n) holds, then κ can take the values 2, 0, -2, -4, The case $\kappa = 2$ is called the rigid one.*

Definition 11 *A multiplicity vector (MV) is a vector whose components are non-negative integers whose sum is n . Further in the text components of the MVs are the multiplicities of the eigenvalues of $n \times n$ -matrices.*

Remark 12 *For a diagonalizable conjugacy class C with MV equal to (m_1, \dots, m_s) one has $d(C) = n^2 - m_1^2 - \dots - m_s^2$.*

Definition 13 *For a given JNF $J^n = \{b_{i,l}\}$ define its corresponding diagonal JNF J^n . A diagonal JNF is a partition of n defined by the multiplicities of the eigenvalues. For each l $\{b_{i,l}\}$ is a partition of $\sum_{i \in I_l} b_{i,l}$ and J' is the disjoint sum of the dual partitions. Thus if for each fixed l one has $b_{1,l} \geq \dots \geq b_{s_l,l}$, then the eigenvalue $l \in L$ is replaced by $b_{1,l}$ new eigenvalues $h_{1,l}, \dots, h_{b_{1,l},l}$ (hence, J^n has $\sum_{l \in L} b_{1,l}$ distinct eigenvalues).*

Remarks 14 *One has the following properties of corresponding JNFs (see [Ko2]):*

- 1) *For l fixed, set g_k for the multiplicity of the eigenvalue $h_{k,l}$. Then the first $b_{s_l,l}$ numbers g_k equal s_l , the next $b_{s_l-1,l} - b_{s_l,l}$ equal $s_l - 1$, ..., the last $b_{1,l} - b_{2,l}$ equal 1.*
- 2) *There hold the equalities $r(J^n) = r(J'^n)$ and $d(J^n) = d(J'^n)$.*
- 3) *To each diagonal JNF there corresponds a unique JNF with a single eigenvalue.*

Lemma 15 *Given the $p+1$ diagonalizable conjugacy classes c_j or C_j satisfying condition (β_n) and Convention 6, condition (α_n) does not hold for them only in*

Case A) : $p = 2$, $n \geq 4$ is even and the MVs of c_2 and c_3 (resp. of C_2 and C_3) both equal $(n/2, n/2)$.

The lemma is proved at the end of the subsection.

Remark 16 *Making use of Definition 13 and Remarks 14 one can extend the lemma to the case of not necessarily diagonalizable matrices (except A_1 or M_1). In such a context, in Case A) each conjugacy class c_2 , c_3 or C_2 , C_3 is either diagonalizable and as in the lemma or with a single eigenvalue and $n/2$ Jordan blocks of size 2 belonging to it. Indeed, this is the only non-diagonal JNF corresponding to the one with two eigenvalues each of multiplicity $n/2$.*

The first important result in the resolution of the DSP was the following

Theorem 17 (*C.Simpson, see [Si1]*) *For generic eigenvalues and under Convention 6 conditions (α_n) and (β_n) together are necessary and sufficient for the solvability of the DSP for given conjugacy classes C_j .*

The same result for classes c_j is proved in [Ko4], Theorem 19. For arbitrary eigenvalues there holds the following theorem (see [Ko3], Theorem 6).

Theorem 18 *Under Convention 6 conditions (α_n) and (β_n) together are necessary and sufficient for the solvability of the weak DSP for given conjugacy classes c_j or C_j .*

Remarks 19 1) In [Si1] C.Simpson has considered the rigid case for diagonalizable matrices and under Convention 6. He has shown that conditions (α_n) and (β_n) together hold only if $p = 2$ and the MVs of the three matrices correspond to one of the four cases:

$(1, \dots, 1)$	$(1, \dots, 1)$	$(n-1, 1)$	hypergeometric family
$(1, \dots, 1)$	$(\frac{n}{2}, \frac{n}{2} - 1, 1)$	$(\frac{n}{2}, \frac{n}{2})$	even family
$(1, \dots, 1)$	$(\frac{n-1}{2}, \frac{n-1}{2}, 1)$	$(\frac{n+1}{2}, \frac{n-1}{2})$	odd family
$(1, 1, 1, 1, 1, 1,)$	$(2, 2, 2)$	$(4, 2)$	extra case

Observe that in all four cases one has $r_2 + r_3 = n$, i.e. there is an equality in condition (β_n) . Although C.Simpson considers only matrices M_j , the result is automatically extended to the case of matrices A_j .

2) If one wants to get rid of the condition the matrices to be diagonalizable (except A_1 or M_1), then to the above list one should add all cases when a diagonal JNF from the list is replaced by a JNF corresponding to it. All JNFs corresponding to the one with n distinct eigenvalues are the ones in which to each eigenvalue there belongs a single Jordan block. Using the notation from Definition 7, give the list of all JNFs corresponding to the other diagonal JNFs (defined by the MVs) encountered in part 1) of the present remarks:

$(n-1, 1)$	$\{2, 1, \dots, 1\}$	
$(\frac{n}{2}, \frac{n}{2} - 1, 1)$	$\{3, 2, \dots, 2, 1\}$	or $\{\{1, \dots, 1\}\{2, 1, \dots, 1\}\}$ ($n/2$ and $n/2 - 2$ units)
$(\frac{n}{2}, \frac{n}{2})$	$\{2, \dots, 2\}$	
$(\frac{n-1}{2}, \frac{n-1}{2}, 1)$	$\{3, 2, \dots, 2\}$	or $\{\{1, \dots, 1\}\{2, 1, \dots, 1\}\}$ ($(n-1)/2$ and $(n-3)/2$ units)
$(\frac{n+1}{2}, \frac{n-1}{2})$	$\{2, \dots, 2, 1\}$	
$(2, 2, 2)$	$\{3, 3\}$	or $\{\{2, 2\}\{1, 1\}\}$
$(4, 2)$	$\{2, 2, 1, 1\}$	

Proof of Lemma 15:

1⁰. Suppose first that one has

$$r_j \leq n/2 \text{ for } j = 2, \dots, p+1 \quad (*)$$

Then one has $d_j \geq 2r_j(n - r_j)$ and there is equality if and only if the MV of c_j or C_j equals $(r_j, n - r_j)$. This follows from Remark 12.

For $r_2 + \dots + r_{p+1}$ fixed the sum $d_2 + \dots + d_{p+1}$ is minimal for $r_2 = r_3 = [n/2]$ where $[.]$ stands for the entire part of. Indeed, one has $d_2 + \dots + d_{p+1} = (r_2 + \dots + r_{p+1})n - r_2^2 - \dots - r_{p+1}^2$ and one has to maximize $r_2^2 + \dots + r_{p+1}^2$ for $r_2 + \dots + r_{p+1}$ fixed while respecting condition $(*)$.

If n is even and $r_2 = r_3 = n/2$, $r_j = 0$ for $j > 3$, then condition (α_n) fails if and only if $n \geq 4$ (this is Case A)); if $r_4 \neq 0$, then condition (α_n) holds. If n is odd, then the sum $d_2 + \dots + d_{p+1}$ is minimal for $r_2 = r_3 = [n/2]$, $r_4 = 1$ and condition (α_n) holds. One cannot have $r_j = 0$ for all $j > 3$ because then condition (β_n) does not hold.

2^0 . Suppose that $r_2 > n/2$. Denote the MV of the class c_2 or C_2 by (m_1, \dots, m_s) , with $m_1 \geq \dots \geq m_s$. Then d_2 is minimal if $m_1 = m_2 = \dots = m_{s-1} = n - r_2$, see Remark 12. The sum $d_3 + \dots + d_{p+1}$ is minimal if $r_3 = m_1 = n - r_2$, $r_4 = \dots = r_{p+1} = 0$ and the MV defining the class c_3 or C_3 equals $(r_2, n - r_2)$.

Set $n = (s - 1)m_1 + m_s$. Recall that $1 \leq m_s \leq m_1$. Hence,

$$d_1 = n^2 - n, \quad d_2 = n^2 - (s - 1)m_1^2 - m_s^2 \geq n^2 - m_1 n, \quad d_3 = 2m_1(n - m_1) \quad \text{and}$$

$$d_1 + d_2 + d_3 \geq 2n^2 - n + m_1 n - 2m_1^2 \geq 2n^2 - n + n - 2 = 2n^2 - 2 \quad \text{because } 1 \leq m_1 < n/2$$

The lemma is proved. \square

1.3 The new results

Definition 20 *The eigenvalues of the matrices A_j or M_j are called k -generic, $k \in \mathbf{N}$, if they satisfy non-genericity relations only with $N \geq k$, see Definition 3 and part 2) of Remarks 4.*

Theorem 21 *Under Convention 6, if the eigenvalues are 2-generic, and if $\kappa \leq 0$ (see Definition 10), then conditions (α_n) and (β_n) are necessary and sufficient for the solvability of the DSP.*

The theorem is proved in Section 2. Examples 29 and 30 below show that the theorem cannot be made stronger.

Theorem 22 *Under Convention 6 and for arbitrary eigenvalues, if $r_2 + \dots + r_{p+1} \geq n + 1$, then the DSP is solvable for such conjugacy classes.*

The theorem is proved in Section 3. Example 29 below shows that for $r_2 + \dots + r_{p+1} = n$ Theorem 22 is no longer true.

Remark 23 *The above two theorems imply that under Convention 6 the weak DSP is solvable but the DSP is not only if $r_2 + \dots + r_{p+1} = n$ and either $\kappa = 2$ or the eigenvalues satisfy a 1-relation.*

Corollary 24 *Under Convention 6 a block upper-triangular tuple of diagonalizable matrices A_j or M_j with 3-generic eigenvalues can be deformed into one from the same conjugacy classes and with trivial centralizer.*

Indeed, 3-genericity implies that for each diagonal block (say, of size $s \geq 3$) there holds condition (β_s) and Case A) from Lemma 15 is avoided; hence, condition (β_n) holds for the tuple of conjugacy classes (the quantity r computed for the whole matrix is not smaller than the sum of the quantities r computed for the diagonal blocks), and Case A) is avoided (because the blocks are of size ≥ 3 – we leave the details for the reader). Hence, for the given tuple of conjugacy classes there hold conditions (α_n) and (β_n) (see Lemma 15). The claim follows now from Lemma 24 from [Ko3]. \square

Corollary 25 *Under Convention 6, if the eigenvalues are 2-generic, and if Case A) is avoided, then for such a block upper-triangular tuple of diagonalizable matrices A_j or M_j there hold conditions (β_n) and (α_n) . Moreover, the tuple can be deformed into one from the same conjugacy classes and with trivial centralizer.*

The first claim is proved as Corollary 24, the second follows from Lemma 24 from [Ko3].

Notation 26 *For a tuple of matrices A_j or M_j in block upper-triangular form $\begin{pmatrix} P_j & Q_j \\ 0 & R_j \end{pmatrix}$ (where $P_j \in gl(l, \mathbf{C})$, $R_j \in gl(n-l, \mathbf{C})$) set $d_j^1 = d(P_j)$, $r_j^1 = r(P_j)$, $d_j^2 = d(R_j)$, $r_j^2 = r(R_j)$, $s_j = \dim \mathcal{X}_j$ where $\mathcal{X}_j = \{Z \in M_{l, n-l} | Z = P_j X_j - X_j R_j, X_j \in M_{l, n-l}\}$. Denote by \mathcal{P} , \mathcal{R} the representations defined by the tuples of matrices P_j , R_j .*

Remark 27 *If the MVs of the diagonalizable matrices P_j and R_j equal respectively (m'_1, \dots, m'_s) , (m''_1, \dots, m''_s) (there might be zeros among these numbers as some eigenvalue might be absent in P_j or R_j), then $s_j = l(n-l) - \sum_{i=1}^s m'_i m''_i$. This implies that if one exchanges the positions of the blocks P_j and R_j , then the quantities s_j do not change.*

Lemma 28 *If the representations \mathcal{P} and \mathcal{R} are with trivial centralizers, then one has*

$$\delta := \dim \text{Ext}^1(\mathcal{P}, \mathcal{R}) = s_1 + \dots + s_{p+1} - 2l(n-l) .$$

Proof:

Notice first that \mathcal{X}_j is the space of right upper blocks of matrices of the form

$$\begin{pmatrix} I & X_j \\ 0 & I \end{pmatrix}^{-1} \begin{pmatrix} P_j & 0 \\ 0 & R_j \end{pmatrix} \begin{pmatrix} I & X_j \\ 0 & I \end{pmatrix} .$$

To obtain δ one must first subtract $l(n-l)$ from $\sum_{j=1}^{p+1} \dim \mathcal{X}_j$ (because the sum of these right upper blocks must be 0) and then again subtract $l(n-l)$ (to factor out the simultaneous conjugation with matrices $\begin{pmatrix} I & X \\ 0 & I \end{pmatrix}$; as A_1 or M_1 is with distinct eigenvalues, no such matrix with $X \neq 0$ commutes with all matrices from the tuple). \square

Example 29 *Consider under Convention 6 a tuple of diagonalizable conjugacy classes c_j for which $r_2 + \dots + r_{p+1} = n$, $n > 2$. Denote by μ_1 an eigenvalue of c_1 and by μ_2, \dots, μ_{p+1} eigenvalues of c_2, \dots, c_{p+1} of maximal possible multiplicity; we assume these multiplicities to be $> n/2$. Suppose that the eigenvalues of the classes c_j satisfy the only non-genericity relation $\mu_1 + \dots + \mu_{p+1} = 0$.*

Denote by $c'_j \subset gl(n-1, \mathbf{C})$ the conjugacy classes obtained from c_j by deleting the eigenvalues μ_j . Hence, condition (β_{n-1}) holds for the classes c'_j and the sum of their eigenvalues is 0. Moreover, the classes c'_j do not correspond to Case A) from Lemma 15 (we let the reader check this oneself).

Hence, there exist block upper-triangular matrices $A_j = \begin{pmatrix} A'_j & D_j \\ 0 & \mu_j \end{pmatrix}$, $A'_j \in c'_j$, whose tuple defines a semi-direct sum (but not a direct one); the matrices A'_j define an irreducible representation. Indeed, one checks directly that $\dim \text{Ext}^1(A, M) = 1$ (this results from $r_2 + \dots + r_{p+1} = n$). The same equality shows that the variety \mathcal{V} consisting of tuples of matrices $A_j \in c_j$ which are

block upper-triangular up to conjugacy (i.e. like A_j above) is of dimension $\dim \mathcal{W}$ where \mathcal{W} is the variety of tuples with trivial centralizers from the classes c_j .

This means that there exist no irreducible tuples from the classes c_j . Indeed, should they exist, their variety (which is part of \mathcal{W}) should contain in its closure the variety \mathcal{V} (see Theorem 6 from [Ko3]), hence, one would have $\dim \mathcal{V} < \dim \mathcal{W}$ which is a contradiction.

The example shows that Theorem 21 is not true without the condition the eigenvalues to be 2-generic and that Theorem 22 is not true if there is an equality in (β_n) .

A similar example can be given for matrices M_j .

Example 30 *There exist triples of diagonalizable 2×2 -matrices M_j^1 (resp. M_j^2) with (generic) eigenvalues equal to (a, b) , (μ, ν) , (η, ξ) (resp. to (c, d) , (μ, ν) , (η, ζ)); same (different) letters denote same (different) eigenvalues.*

Then there exists a block upper-triangular triple of matrices $M_j = \begin{pmatrix} M_j^1 & B_j \\ 0 & M_j^2 \end{pmatrix}$ defining a semi-direct sum of the representations \mathcal{P}^1 and \mathcal{P}^2 defined by the matrices M_j^1 and M_j^2 (because $\dim \text{Ext}^1(\mathcal{P}^1, \mathcal{P}^2) = 1$).

One checks directly that

- a) the centralizer of the matrices M_j is trivial;*
- b) their eigenvalues can be chosen 2-generic (we assume that they satisfy only the following non-genericity relations: $ab\mu\nu\eta\xi = 1$ and $cd\mu\nu\eta\zeta = 1$);*
- c) one has $\kappa = 2$ for the triple of conjugacy classes of the matrices M_j .*

As $\kappa = 2$, one cannot have coexistence of irreducible and reducible triples, see [Ka]. This means that the DSP is not solvable for the triple of conjugacy classes of the matrices M_j (but the weak DSP is, see a)). Hence, Theorem 21 is not true for $\kappa = 2$.

A similar example can be given for matrices A_j .

2 Proof of Theorem 21

2.1 The method of proof

¹⁰. Suppose that for the conjugacy classes c_j or C_j (with 2-generic eigenvalues) there hold conditions (α_n) and (β_n) . The variety of matrices $A_j \in c_j$ (satisfying (1)) or of matrices $M_j \in C_j$ (satisfying (2)) is of dimension $d' := d_1 + \dots + d_{p+1} - n^2 + 1$ at each tuple with trivial centralizer, see [Ko6], Proposition 2.

Given a reducible tuple of matrices from these conjugacy classes (block upper-triangular up to conjugacy, with trivial centralizer, with given sizes of the diagonal blocks and with given conjugacy classes of the restrictions of the matrices to the diagonal blocks) we compute the dimension d'' of the variety of such tuples and we show that $d'' < d'$. If this is the case of all such reducible tuples, then the variety of tuples with trivial centralizers must contain irreducible tuples as well. Hence, the DSP is solvable for the given conjugacy classes.

Lemma 31 *Under Convention 6, suppose that the tuple of diagonalizable matrices A_j or M_j is as in Notation 26, and that the representations \mathcal{P} and \mathcal{R} are with trivial centralizers. If $\delta := \dim \text{Ext}^1(\mathcal{P}^1, \mathcal{P}^2) > 1$, then $d'' < d'$.*

All lemmas from the proof of the theorem are proved in Subsection 2.4.

Corollary 32 *If the representations \mathcal{P} and \mathcal{R} from the lemma are irreducible, then there exist irreducible tuples from the conjugacy classes $c(P_j) \times c(R_j)$.*

The corollary is immediate.

We prove the theorem for diagonalizable matrices in $2^0 - 5^0$ and then we treat the general case in $6^0 - 11^0$.

2.2 The proof for diagonalizable matrices

2^0 . Prove the theorem for diagonalizable matrices.

Lemma 33 *Suppose that the tuples of diagonalizable matrices $P_j \in gl(l, \mathbf{C})$ and $R_j \in gl(n-l, \mathbf{C})$ (resp. $P_j \in GL(l, \mathbf{C})$ and $R_j \in GL(n-l, \mathbf{C})$) are with trivial centralizers, P_1 and R_1 being each with distinct eigenvalues and with no eigenvalue in common, and that $l \geq n-l \geq 2$. Then $\delta \geq 2$ with the exception of the cases listed below¹. In all of them one has $p = 2$. (We give the list of the eigenvalues of the matrices P_2, R_2 and P_3, R_3 , equal (different) letters denote equal (different) eigenvalues if they correspond to one and the same index j . In Cases C) – F) one can exchange the roles of P_2, R_2 and P_3, R_3 .)*

Case B)	$l = n - l = 2$	(a, b) (a, b)	(c, d) (c, d)
Case C)	$l = n - l = 2$	(a, b) (a, g)	(c, d) (c, d)
Case D)	$l = n - l = 3$	(a, b, c) (a, b, c)	(f, g, g) (f, g, g)
Case E)	$l = 2q + 1, n - l = 2$	$(\underbrace{a, \dots, a}_{q \text{ times}}, \underbrace{b, \dots, b}_{q \text{ times}}, c)$ (a, b)	$(\underbrace{f, \dots, f}_{q+1 \text{ times}}, \underbrace{g, \dots, g}_{q \text{ times}})$ (f, g)
Case F)	$l = 2q, n - l = 2$	$(\underbrace{a, \dots, a}_{q \text{ times}}, \underbrace{b, \dots, b}_{q-1 \text{ times}}, c)$ (a, b)	$(\underbrace{f, \dots, f}_{q \text{ times}}, \underbrace{g, \dots, g}_{q \text{ times}})$ (f, g)

In Case B) condition (α_n) does not hold for the conjugacy classes $C(P_j) \times C(R_j)$, in the other cases it holds and is an equality. One has $\delta = 0$ in Case B) and $\delta = 1$ in Cases C) – F).

Corollary 34 *In the conditions of the lemma and if the representations \mathcal{P} and \mathcal{R} are irreducible the DSP is solvable for the tuple of conjugacy classes $C(S_j) = C(P_j) \times C(R_j)$ (except for Cases B) – F)).*

Proof:

The condition $\delta > 0$ implies that there exists a semi-direct sum of the representations \mathcal{P} and \mathcal{R} (we use Notation 26 here) which is not reduced to a direct one. The centralizer of this semi-direct sum is trivial. Indeed, one can assume that P_1 and R_1 are diagonal, so a matrix X from the centralizer must be also diagonal. The P -block of X commutes with all matrices P_j , hence, it is scalar (because the centralizer of \mathcal{P} is trivial). In the same way the R -block of X must be scalar. Finally, these blocks must be equal, otherwise the commutation relations imply that all blocks Q_j must be 0 which contradicts the sum of \mathcal{P} and \mathcal{R} not to be a direct one.

¹When listing the cases we begin with B, not with A, in order to avoid mixing up with Case A) from Lemma 15

Hence, the variety \mathcal{V} of tuples of matrices defining semi-direct sums of \mathcal{P} and \mathcal{R} is non-empty and its dimension is smaller than the dimension of the variety $\mathcal{W} \supset \mathcal{V}$ of tuples with trivial centralizers of matrices from the classes $C(S_j)$ (see Lemma 31). Hence, \mathcal{V} is locally a proper subvariety of \mathcal{W} and a tuple from \mathcal{V} can be deformed into a tuple from $\mathcal{W} \setminus \mathcal{V}$ (see Theorem 6 from [Ko3]). The latter must be irreducible. Indeed, \mathcal{V} contains locally all reducible tuples because \mathcal{P} and \mathcal{R} are irreducible. \square

3⁰. Deduce the theorem from the corollary. The weak DSP is solvable for conjugacy classes in the conditions of the theorem. Indeed, 2-genericity implies that a tuple from the given conjugacy classes is (up to conjugacy) block upper-triangular with diagonal blocks all of sizes ≥ 2 and defining irreducible representations. (We assume that there is more than one diagonal block, otherwise the tuple is irreducible and there is nothing to prove.)

The restriction of the tuple to the union of diagonal blocks is a tuple from the same conjugacy classes (because the conjugacy classes are diagonalizable). Consider a couple of consecutive diagonal blocks. (We denote the restrictions of the matrices A_j or M_j to these two blocks by $A_j^i, M_j^i, i = 1, 2$.) They are both of size ≥ 2 , and if one is not in one of the Cases B) – F), then one can apply the above corollary and obtain the existence of irreducible tuples of matrices from the conjugacy classes $C(A_j^1) \times C(A_j^2)$ (resp. $C(M_j^1) \times C(M_j^2)$). Thus we obtain a block-diagonal tuple of $n \times n$ -matrices with one diagonal block less. Continuing like this we end with an irreducible tuple of matrices which solves the DSP for the conjugacy classes c_j or C_j .

4⁰. There might be a problem, however, with Cases B) – F). First of all notice that this does not happen if $p \geq 3$. Indeed, in this case one can always choose two diagonal blocks defining irreducible representations and in which at least four conjugacy classes $C(A_j^1) \times C(A_j^2)$ (resp. $C(M_j^1) \times C(M_j^2)$) are not scalar (including $j = 1$). So one can permute the diagonal blocks (to get two consecutive blocks not from Cases B) – F)) and the proof is carried out as in 3⁰.

5⁰. So suppose that $p = 2$. We start again with the restriction of the tuple to the set of diagonal blocks defining irreducible representations. It is not possible to have all couples of diagonal blocks to correspond to Case B) from the lemma because this will mean that the classes c_j or C_j are from Case A) of Lemma 15. So choose a couple of consecutive diagonal blocks which are not from Case B) and replace them by a single block B defining a semi-direct sum of the representations which they define while keeping the other diagonal blocks the same. This is possible because for the chosen blocks one has $\delta \geq 1$, see the lemma.

At each next step one has a block-diagonal tuple with diagonal blocks defining irreducible representations except B which defines one with trivial centralizer. At each step choose a block W different from B and next to B (hence, their couple is not from Case B) because B is of size > 2), so one can replace it by a new block (which is the new block B) defining a semi-direct sum of the representations they define. So at each step the blocks B, W are not from Case B).

At the last step we obtain a representation with trivial centralizer. The last couple of blocks B, W is not from Cases B) – F). Indeed, should it be from these cases, then for the conjugacy classes c_j or C_j one should have $\kappa \geq 2$ (to be checked directly).

Hence, for the last couple of blocks B, W one has $\delta \geq 2$. This means that $d'' < d'$, see 1⁰. This proves the theorem in the case of diagonalizable matrices.

2.3 The proof in the general case

6⁰.

Convention 35 *From here till the end of this subsection when Case A) of Lemma 15 or Cases*

B) – F) of Lemma 33 are cited the JNFs of the matrices A_j or M_j ($j \geq 2$) will be assumed either to be the ones given in these two lemmas or to correspond to them, see Remarks 16 and 19.

Such a change of the definition of these cases does not change the quantity δ , see part 2) of Remarks 14. Hence, Lemma 33 is applicable after the change as well.

7⁰. Consider a tuple in block upper-triangular form whose diagonal blocks define irreducible representations. Consider the restriction of the tuple to the set of diagonal blocks. The conjugacy class c'_j (resp. C'_j) of the restriction of the matrix A_j (resp. M_j) from the tuple to the set of diagonal blocks belongs to the closure of c_j (resp. of C_j) but is not necessarily equal to it (one might obtain a “less generic” Jordan structure when cutting off the blocks above the diagonal; the eigenvalues and their multiplicities do not change). If for the conjugacy classes c'_j or C'_j the index of rigidity is ≤ 0 , then as in the case of diagonalizable conjugacy classes one shows that the DSP is solvable for the classes c'_j or C'_j . This implies its solvability for the classes c_j (resp. C_j) (which can be proved by analogy with part 2 of Lemma 53 from [Ko2]).

8⁰. Suppose (in 8⁰ – 11⁰) that the index of rigidity of the tuple of conjugacy classes c'_j or C'_j is > 0 . Then for some $j_0 > 1$ there exists a conjugacy class c''_{j_0} (or C''_{j_0} ; we write further only c''_{j_0} for short) such that

- 1) c'_{j_0} belongs to the closure of c''_{j_0} ;
- 2) c''_{j_0} is obtained from c'_{j_0} when a couple of Jordan blocks with one and the same eigenvalue, of sizes l, s , $l \geq s$, are replaced by Jordan blocks (with the same eigenvalues) of sizes $l+1, s-1$, see Section 8 in [Ko2]; the rest of the Jordan structure remains the same;
- 3) c''_{j_0} belongs to the closure of c_{j_0} (eventually, $c''_{j_0} = c_{j_0}$).

When passing from c'_{j_0} to c''_{j_0} the index of rigidity decreases by at least 2. If the change 2) can take place by changing the JNF of the restriction of A_{j_0} or M_{j_0} to some diagonal block, then we perform this change and further the proof is done as in the case of diagonalizable matrices.

9⁰. If for the change 2) one has to change a block above the diagonal, and if there are at least 3 diagonal blocks, then one proceeds as in 5⁰ and one proves that $d'' < d'$ exactly in the same way.

Indeed, at the first step one replaces two diagonal blocks (defining irreducible representations) by a single one (defining their semi-direct sum). Namely, using Notation 26, one chooses the block Q_{j_0} such that the change 2) to take place. Then one chooses the block Q_1 such that condition (1) or (2) to hold (recall that A_1 and M_1 are with distinct eigenvalues, therefore changing the block Q_1 while keeping P_1 and R_1 the same does not change the conjugacy class of A_1 or M_1).

The next steps are as in 5⁰.

10⁰. If there are just two diagonal blocks, not from Case B), then one first constructs a block upper-triangular tuple (with trivial centralizer) defining a semi-direct sum of the representations defined by the diagonal blocks but without changing the class c'_{j_0} .

Then conjugate the tuple with a block upper-triangular matrix so that the matrix A_{j_0} or M_{j_0} to be in JNF (hence, it will be block diagonal as well). After this perform a change $A_{j_0} \mapsto A_{j_0} + \varepsilon U$ or $M_{j_0} \mapsto M_{j_0} + \varepsilon U$, $\varepsilon \in (\mathbf{C}, 0)$ where only the left lower block of U is non-zero and is not of the form $R_{j_0}X - XP_{j_0}$; U is chosen such that for $\varepsilon \neq 0$ one has $A_{j_0} \in c''_{j_0}$ (resp. $M_{j_0} \in C''_{j_0}$).

To preserve condition (1) or (2) one looks then for deformations of the matrices A_j or M_j , $j \neq j_0$, analytic in ε . Such a deformation exists, see the description of the “basic technical tool” in [Ko2] (one conjugates the matrices A_j or M_j , $j \neq 0$, with matrices which are analytic deformations of I).

Lemma 36 *For $\varepsilon \neq 0$ small enough the constructed tuple is irreducible.*

The lemma implies the theorem in this case.

11⁰. If the two diagonal blocks are from Case B), then one change 2) is not sufficient to make the index of rigidity ≤ 0 . Hence, at least two changes are necessary. With the first of them we construct the semi-direct sum of representations defined by the two diagonal blocks; this time we change one of the JNFs for $j = j_* > 1$. When performing this change we change the block Q_{j_*} and then we change Q_1 to restore condition (1) or (2).

Suppose that the second change must take place for $j = j_0 \neq j_*$. Then after the second change 2) (performed as in 10⁰, using an analytic deformation) one has an irreducible representation by full analogy with Lemma 36.

If $j_* = j_0$ (and, say, $j_0 = 2$), then there are two possibilities. Either this JNF has a single eigenvalue, or it is with two double eigenvalues and three Jordan blocks. In the first case one can assume that the couple A_2, U (resp. M_2, U) looks like this (after the analog of the conjugation from 10⁰):

$$A_2 = \begin{pmatrix} a & 1 & 0 & 0 \\ 0 & a & 0 & \underline{1} \\ 0 & 0 & a & 1 \\ 0 & 0 & 0 & a \end{pmatrix}, \quad U = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We underline the unit which is introduced after the first change 2). Its introduction results in changing the JNF like this: $\{2, 2\} \mapsto \{3, 1\}$. In the second case the couple looks like this:

$$A_2 = \begin{pmatrix} a & 0 & \underline{1} & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & b \end{pmatrix}, \quad U = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

For the rest the proof is carried out as in 8⁰ – 10⁰. The theorem is proved. \square

2.4 Proofs of the lemmas

Proof of Lemma 31:

To obtain d'' one must add $l(n-l)$ to d''' , the dimension of the variety of block upper-triangular tuples as in the lemma (truly block upper-triangular, not only up to conjugacy). Indeed, $l(n-l)$ is the size of the left lower block and adding this corresponds to taking into account the possibility to conjugate such a tuple by matrices of the form $\begin{pmatrix} I & 0 \\ X & I \end{pmatrix}$.

One has $d''' = \Delta_1 + \Delta_2 + \Delta_3$ where $\Delta_1 = \sum_{j=1}^{p+1} d_j^1 - l^2 + 1$, $\Delta_2 = \sum_{j=1}^{p+1} d_j^2 - (n-l)^2 + 1$ and $\Delta_3 = \sum_{j=1}^{p+1} s_j - l(n-l)$ (the contributions to d''' from the P -, R - and Q -block).

On the other hand, $d_j = d_j^1 + d_j^2 + 2s_j$ (this can be deduced from Remark 12). Hence, $d''' = \sum_{j=1}^{p+1} d_j - \sum_{j=1}^{p+1} s_j - n^2 + l(n-l) + 2 = \sum_{j=1}^{p+1} d_j - \delta - n^2 - l(n-l) + 2$ and $d'' = \sum_{j=1}^{p+1} d_j - \delta - n^2 + 2$.

One has $d' = \sum_{j=1}^{p+1} d_j - n^2 + 1 = d'' + \delta - 1$. Hence, for $\delta > 1$ one has $d' > d''$. \square

Proof of Lemma 33:

We transform the proof of the lemma into finding the cases when $\delta \leq 1$.

Statement 37 One has $s_j \geq r_j^1(n-l)$ (A) and $s_j \geq r_j^2 l$ (B) (see Notation 26).

Proof:

Use Remark 27 (and the notation from it) and Lemma 28. Denote by μ' (resp. μ'') the biggest among the numbers m'_j (resp. m''_j). Then $s_j \geq l(n-l) - \mu'(n-l) = r_j^1(n-l)$ because $\sum_{i=1}^s m'_i m''_i \leq \mu' \sum_{i=1}^s m''_i = \mu'(n-l)$. In the same way $s_j \geq l(n-l) - \mu'' l = r_j^2 l$. \square

Remark 38 Inequality (A) becomes an equality exactly if $m''_i = 0$ whenever $m'_i < \mu'$. Inequality (B) becomes an equality exactly if $m'_i = 0$ whenever $m''_i < \mu''$.

Statement 39 If for some index $j > 1$ (say, $j = 2$) one has $r_j^1 = 0$, $r_j^2 > 0$, then one has $\delta \geq 2$. The same is true if $r_j^1 = r_j^2 = 0$ and c_j is not scalar. The same is true if $r_j^1 > 0$, $r_j^2 = 0$.

Proof:

Consider the first and the second of the three claims. By (A) one has $s_3 + \dots + s_{p+1} \geq (r_3^1 + \dots + r_{p+1}^1)(n-l) \geq l(n-l)$; recall that $s_1 = l(n-l)$. In the first claim one has also $s_2 \geq r_j^2 l \geq 2$, hence, $\delta \geq 2$. In the second claim the conjugacy class c_2 defines the MV $(l, n-l)$ and one has $s_2 = l(n-l) \geq 2$ and again $\delta \geq 2$. The third claim is proved in the same way as the first one using (B). \square

Convention 40 From now till the end of the proof of the lemma we assume (using the above statement) that for all indices $j > 1$ one has $r_j^1 > 0$, $r_j^2 > 0$.

Statement 41 If $p \geq 3$, then $\delta \geq 2$.

Proof:

It suffices to consider the following two cases (up to permutation of the indices $j > 1$):

- 1) $r_2^1 \geq l/2$, $r_3^1 \geq l/2$, $r_4^1 > 0$;
- 2) $r_2^1 > 0$, $l/2 > r_j^1 > 0$ for $j > 2$.

In case 1) one has $s_2 + s_3 \geq l(n-l)$ (see (A)), $s_4 \geq n-l \geq 2$, $s_1 = l(n-l)$, so $\delta \geq 2$, see Lemma 28.

In case 2) recall first that $r_j^2 > 0$ for $j > 2$. For $j = 3, 4, \dots, p+1$ one has $s_j > r_j^1(n-l)$, i.e. $s_j \geq r_j^1(n-l) + 1$, see Statement 37 and Remark 38. One has $s_1 = l(n-l)$, $s_2 \geq r_2^1(n-l)$ (see (A)), hence, $s_1 + \dots + s_{p+1} \geq 2l(n-l) + 2$ and again $\delta \geq 2$. \square

Convention 42 From now till the end of the proof of the lemma we assume that $p = 2$, see Statement 41.

Statement 43 If $r_2^1 + r_3^1 \geq l+1$ or $r_2^2 + r_3^2 \geq n-l+1$, then $\delta \geq 2$.

Indeed, if $r_2^1 + r_3^1 \geq l+1$, then (see (A)) $s_2 + s_3 \geq (l+1)(n-l) \geq l(n-l) + 2$ and $\delta \geq 2$. In the same way if $r_2^2 + r_3^2 \geq n-l+1$, then $s_2 + s_3 \geq l(n-l+1) \geq l(n-l) + 2$ and $\delta \geq 2$. \square

Statement 44 If l is even and $r_2^1 = r_3^1 = l/2$, then $\delta \geq 2$, except in Cases B), C) and F) from the lemma.

Proof:

1⁰. If $l = 2$, then $n - l = 2$ and one has $\delta \leq 1$ only in one of Cases B) or C) from the lemma.

2⁰. If $l \geq 4$, then $\delta \geq 2$. Indeed, to avoid Case A) from Lemma 15 for the block P , one must suppose that at least one of the two matrices P_2 and P_3 (say, P_2) has at least three distinct eigenvalues. Assume that the MV of P_2 looks like this: (m'_1, \dots, m'_s) , with $m'_1 = \mu' > m'_2 \geq \dots \geq m'_s$ (the inequality $m'_1 > m'_2$ results from $r_2^1 = l/2$).

If for at least two indices $i > 1$ one has $m''_i \neq 0$, then for them one has $m'_i m''_i < \mu' m''_i$ and $\sum_{i=1}^s m'_i m''_i \leq \mu' \sum_{i=1}^s m''_i - 2 = \mu'(n - l) - 2$. Hence, $s_2 \geq r_2^1(n - l) + 2$ (see Remark 27), $s_3 \geq r_3^1(n - l)$ (see (A)) and $\delta \geq 2$.

3⁰. If for only one index $i > 1$ one has $m''_i \neq 0$ (i.e. R_2 has only two different eigenvalues), then similarly $s_2 \geq r_2^1(n - l) + 1$ with equality only if the MV of P_2 equals $(l/2, l/2 - 1, 1)$ and the two eigenvalues, of the two greatest multiplicities, are eigenvalues of R_2 as well; moreover, its only eigenvalues.

4⁰. If P_3 has at least three different eigenvalues, then in the same way $s_3 \geq r_3^1(n - l) + 1$ and, hence, $\delta \geq 2$. So the only possibility to have $\delta \leq 1$ is the MV of P_3 to be $(l/2, l/2)$. If $n - l = 2$, then $\delta \leq 1$ only in Case F). If $n - l > 2$, then R_3 must have at least three distinct eigenvalues (otherwise condition (α_{n-l}) fails for the block R) and $s_3 \geq lr_3^2 + 1$. One has also $s_2 \geq lr_2^2 + 1$ (to be checked directly), hence, again $\delta \geq 2$. \square

Statement 45 Suppose that $r_2^1 + r_3^1 = l$. If $r_2^1 > l/2$, $r_3^1 < l/2$ or $r_2^1 < l/2$, $r_3^1 > l/2$, then $\delta \geq 2$ except in Cases D), E) from the lemma.

Proof:

1⁰. Without loss of generality we assume that $r_2^1 > l/2$, $r_3^1 < l/2$. If $l = 3$ and $n - l = 2$ or $n - l = 3$, then one has $\delta \leq 1$ only in Case E) with $q = 1$ or in Case D) of the lemma. Indeed, s_j is minimal only if all eigenvalues of R_j are eigenvalues of P_j as well for $j = 2, 3$.

2⁰. If $l \geq 5$ and $n - l \geq 4$, then $\delta \geq 2$. Indeed, if the MVs of P_3 and R_3 equal respectively (m'_1, \dots, m'_s) , (m''_1, \dots, m''_s) , with $m'_1 = \mu' > m'_2 \geq \dots \geq m'_s$, then one has $m'_i m''_i \leq (\mu' - 1)m''_i$ for $i > 1$ and $m''_i > 0$; hence,

$$\sum_{i=1}^s m'_i m''_i \leq \mu' m''_1 + (\mu' - 1) \sum_{i=2}^s m''_i = \mu' \sum_{i=1}^s m''_i - \sum_{i=2}^s m''_i = \mu'(n - l) - \sum_{i=2}^s m''_i.$$

If $\sum_{i=2}^s m''_i \geq 2$, then $s_3 \geq r_3^1(n - l) + 2$ (see Remark 27), $s_2 \geq r_2^1(n - l)$ (see (A)) and $\delta \geq 2$. So δ can be ≤ 1 only in case that $\sum_{i=2}^s m''_i = 1$, i.e. the MV of R_3 is of the form $(n - l - 1, 1)$. If this is so, then the MV of R_2 is $(1, \dots, 1)$ (otherwise (α_{n-l}) fails for the block R), i.e. R_2 has distinct eigenvalues. Hence, $s_2 \geq l(n - l - 1)$ whatever the eigenvalues of P_2 are.

But then s_3 is minimal if and only if the MV of P_3 equals $(l - 1, 1)$ and P_3 has the same eigenvalues as R_3 (the proof of this is left for the reader). In this case $s_3 = (l - 1) + (n - l - 1) = n - 2$, hence, $s_2 + s_3 \geq l(n - l) + n - l - 2$ and for $n - l \geq 4$ one has $\delta \geq 2$.

3⁰. If $l \geq 5$ and $n - l = 2$, and if P_2 has at least 4 distinct eigenvalues, then $s_2 \geq l + 2$. Indeed, s_2 is minimal only if each eigenvalue of R_2 is eigenvalue of P_2 as well.

In such a case one has $s_2 = 2l - m'_{i_1} - m'_{i_2}$ where m'_{i_1}, m'_{i_2} are the multiplicities of the eigenvalues of R_2 as eigenvalues of P_2 . As $m'_{i_1} + m'_{i_2} \leq l - 2$ (there are at least two more eigenvalues of P_2 , each of multiplicity ≥ 1), one gets $s_2 \geq l + 2$. In a similar way, $s_3 \geq l$, with equality when P_3 has two eigenvalues which are eigenvalues of R_3 as well, hence, $\delta \geq 2$.

If P_2 has exactly three distinct eigenvalues, then one has $s_2 \geq l + 1$ with equality exactly if the eigenvalue which is not eigenvalue of P_2 is simple. Hence, $\delta \leq 1$ only in Case E) from the lemma.

4⁰. If $l \geq 5$ and $n - l = 3$, then at least one of the matrices R_2, R_3 must have 3 distinct eigenvalues (otherwise (β_3) fails for the block R). The respective quantity s_j must be $\geq 2l = r_j^2 l$, see (B). If the other matrix R_j ($j = 2$ or 3) has also 3 distinct eigenvalues, then $s_2 + s_3 \geq 4l > 3l + 2$ and $\delta \geq 2$.

If the MV of the other matrix R_j (say, R_3) equals $(2, 1)$, then s_3 is minimal exactly if P_3 has the same eigenvalues as R_3 , of multiplicities $l - 1$ and 1 . In this case $s_3 = l + 1$. But then P_2 must be with distinct eigenvalues (otherwise (α_l) fails for the block P), $s_2 \geq 3l - 3$, and $\delta > 2$.

5⁰. If $l = 4$, then one can have $r_2^1 > 2$, $r_3^1 < 2$ only if P_3 has four distinct eigenvalues and the MV of P_3 is $(1, 3)$. We let the reader check oneself that in all possible cases ($n - l = 2, 3$ or 4) one has $\delta \geq 2$. \square

The lemma follows from Statements 39, 41, 43, 44 and 45. \square

Proof of Lemma 36:

Denote by \mathcal{T} , the matrix algebra of all block upper-triangular matrices with square diagonal blocks of sizes l and $n - l$. A priori the representation defined by the deformed matrices is either irreducible (and the corresponding matrix algebra is $gl(n, \mathbf{C})$) or is reducible and defines a matrix algebra which up to analytic conjugation equals \mathcal{T} (the statement results from a more general one which can be found in [Ko7]). The second case, however, is impossible because such a conjugation of A_{j_0} or M_{j_0} (with a matrix $I + O(\varepsilon)$) cannot make the left lower block of U disappear (because it is not of the form $R_{j_0}X - XP_{j_0}$). \square

3 Proof of Theorem 22

3.1 Proof in the case of matrices A_j

Definition 46 A conjugacy class is called *regular* if to every eigenvalue there corresponds a single Jordan block of size equal to the multiplicity of the eigenvalue.

Remark 47 The JNFs of all regular conjugacy classes correspond to each other (see Definition 13) and, in particular, to the diagonal JNF with distinct eigenvalues and to the JNF with a single eigenvalue and a single Jordan block of size n .

Proposition 48 The DSP is positively solvable for classes c_j where c_1 is regular and one has $r_2 + \dots + r_{p+1} \geq n + 1$.

The proposition implies the theorem in the case of matrices A_j . To prove the proposition we need the following lemma.

Lemma 49 *The DSP is positively solved for tuples of nilpotent conjugacy classes c_j with $r_1 + \dots + r_{p+1} \geq 2n$ in which $r_1 = n - 1$, i.e. the conjugacy class c_1 has a single Jordan block of size n .*

The lemma is a particular case of the results in [Ko8]. It follows also from the ones in [C-B].

Proof of the proposition:

Given an irreducible tuple of nilpotent matrices A_j satisfying the conditions of the lemma one can deform it analytically into an irreducible tuple of matrices A'_j where for each j either $J(A'_j) = J(A_j)$ or $J(A'_j)$ corresponds to $J(A_j)$. The eigenvalues of the matrices A'_j must be close to 0. These statements can be deduced from [Ko2], see the definition of the basic technical tool there which is a way to deform analytically tuples of matrices with trivial centralizers; compare also with Lemma 53 from [Ko2].

Thus one obtains the positive solvability of the DSP for all tuples of JNFs $J(c_j)$ satisfying the condition $r_2 + \dots + r_{p+1} \geq n + 1$; see Definition 13 and Remarks 14 (especially part 2) of them). However, solvability is proved only for eigenvalues close to 0.

By multiplying the tuples of matrices A'_j by non-zero complex numbers (i.e. $(A'_1, \dots, A'_{p+1}) \mapsto (gA'_1, \dots, gA'_{p+1}), g \in \mathbb{C}^*$) one can obtain irreducible tuples with the same JNFs as A'_j and with any eigenvalues whose sum (taking into account the multiplicities) is 0. This proves the proposition. \square

3.2 Proof for matrices M_j

Suppose that for some conjugacy classes C_j satisfying the conditions of the theorem there exist no irreducible tuples. Then there exist tuples with trivial centralizers. This follows from Theorem 18 and from Lemma 15.

Each such tuple can be conjugated to a block upper-triangular form in which the diagonal blocks define irreducible or one-dimensional representations. Denote by s_1, \dots, s_ν the sizes of the diagonal blocks. We say that these sizes (considered up to permutation) define the *type* of the tuple. The tuple is called *maximal* if there is no tuple with trivial centralizer and of type s'_1, \dots, s'_h such that $h < \nu$ and the sizes s'_i are obtained from the sizes s_j by one or several operations of the form $(s_{j_1}, s_{j_2}) \mapsto s_{j_1} + s_{j_2}$. We say that the type s'_1, \dots, s'_h is *greater* than the type s_1, \dots, s_ν .

Lemma 50 *Given a maximal tuple of matrices M_j one can construct a tuple of matrices $A_j \in c_j$ of the same type, with trivial centralizer, with $M_j = \exp(2\pi i A_j)$ (up to conjugacy) where for $j > 1$ the matrix A_j has no couple of eigenvalues whose difference is a non-zero integer.*

The lemma is proved in the next subsection.

Remark 51 *The condition “ $M_j = \exp(2\pi i A_j)$ (up to conjugacy)” is introduced with the aim to use the fact that the monodromy operators of the Fuchsian system $dX/dt = (\sum_{j=1}^{p+1} A_j/(t - a_j))X$ (**) in the absence of non-zero integer differences between the eigenvalues of the matrices A_j equal (up to conjugacy) $\exp(2\pi i A_j)$. See the definition of the monodromy operators in the Introduction of [Ko2].*

For the tuple of matrices A_j from the lemma one has that they can be analytically deformed into an irreducible tuple of such matrices. Indeed, for their conjugacy classes the DSP is positively solved (this is already proved in Subsection 3.1) and all reducible tuples from these classes belong to the closure of the variety of irreducible tuples, see Theorem 6 from [Ko3].

All irreducible tuples of matrices A_j^0 close to tuples A_j from the lemma define Fuchsian systems $dX/dt = (\sum_{j=1}^{p+1} A_j^0/(t-a_j))X$ (***) whose monodromy groups must be (up to conjugacy) from the type of the tuple of matrices M_j from the lemma. This follows from the tuple of matrices M_j being maximal.

Consider the monodromy operators (denoted also by M_j) of systems (**) with matrices-residua A_j . One has $M_j = \exp(2\pi i A_j)$ (up to conjugacy) and there is a bijection between the eigenvalues of the matrices A_j and the ones of the matrices M_j . For each diagonal block the sum of the eigenvalues of the matrices A_j from the lemma is 0. Hence, the sum of the same eigenvalues of the matrices A_j^0 is also 0. If the monodromy group of system (***) is of the type of the one of system (**), then by Theorem 5.1.2 from [Bo] it should be possible to conjugate the tuple of matrices A_j^0 to a block upper-triangular form with blocks as in the type of the matrices M_j . This contradicts the irreducibility of the tuple of matrices A_j^0 .

Remark 52 *When applying Theorem 5.1.2 from [Bo] we use the fact that there are no non-zero integer differences between the eigenvalues of the matrices A_j . Thus to each eigenvalue σ of M_j of a given multiplicity there corresponds only one eigenvalue λ of A_j (which is of the same multiplicity) where $\sigma = \exp(2\pi i \lambda)$. Theorem 5.1.2 from [Bo] speaks about the exponents (i.e. the eigenvalues of the matrices A_j) corresponding to an invariant subspace. In the absence of non-zero integer differences these exponents are defined by the eigenvalues of the monodromy operators in a unique way.*

The theorem is proved. \square

3.3 Proof of Lemma 50

1⁰. One can construct for each size s_i of the type a tuple of matrices $A_{i,j}^*$ such that one has (up to conjugacy) $\exp(2\pi i A_{i,j}^*) = M_{i,j}^*$ (*) where $M_{i,j}^*$ are the restrictions of the matrices M_j to the diagonal block of size s_i , and the matrices $A_{i,j}^*$ define an irreducible or one-dimensional representation. In the one-dimensional case the claim is evident. In the irreducible case one can construct a Fuchsian system with matrices-residua equal up to conjugacy to $A_{i,j}^*$ (where $A_{i,j}^*$ satisfy (*)) the real parts of whose eigenvalues can be chosen to belong to $[0, 1)$ for $j > 1$ (to avoid non-zero integer differences between eigenvalues); the construction is explained in [ArII].

2⁰. Consider the tuple of matrices A_j' which are block-diagonal their restrictions to each diagonal block of size s_i being equal to the blocks $A_{i,j}^*$ from 1⁰. We complete them (in 3⁰) by adding entries in the blocks above the diagonal (the newly obtained matrices are denoted by A_j) so that one would have $\exp(2\pi i A_j) = M_j$ up to conjugacy. We do this for $j > 1$ and then we define A_1 so that $A_1 + \dots + A_{p+1} = 0$. As A_1 has distinct eigenvalues, whatever entries we add in the blocks above the diagonal, they do not change the conjugacy class of A_1 . As $\exp(2\pi i A_1') = M_1$ up to conjugacy, one will also have $\exp(2\pi i A_1) = M_1$ up to conjugacy.

3⁰. One can conjugate the matrix M_j by a block upper-triangular matrix B_j so that the diagonal blocks of $(B_j)^{-1} M_j B_j$ of sizes s_i to be in JNF and in the blocks above the diagonal non-zero entries to be present only in positions (i, j) such that the i -th and j -th eigenvalues coincide. For each eigenvalue $\sigma_{k,j}$ of M_j denote by $M_j(\sigma_{k,j})$ the matrix whose restriction to the rows and columns of the eigenvalue $\sigma_{k,j}$ are the same as the ones of $(B_j)^{-1} M_j B_j$ and the rest of its entries are 0.

One can conjugate the matrices A'_j by block-diagonal matrices D_j so that the matrix $(D_j)^{-1}A'_jD_j$ to be in JNF and for each diagonal block there to hold $\exp(2\pi i A_{i,j}^*) = M_{i,j}^*$ (up to conjugacy).

Set $(D_j)^{-1}A'_jD_j = \sum_{k,j} \lambda_{k,j} A'_j(\lambda_{k,j})$ where $\lambda_{k,j}$ are the distinct eigenvalues of A'_j and $A'_j(\lambda_{k,j})$ is the matrix whose restriction to the rows and columns of the eigenvalue $\lambda_{k,j}$ is the same as the one of $(D_j)^{-1}A'_jD_j$ and the rest of its entries are 0. Define the matrices $A_j(\lambda_{k,j})$ by analogy with the matrices $A'_j(\lambda_{k,j})$.

Recall that one has $\sigma_{k,j} = \exp(2\pi i \lambda_{k,j})$. Hence, for each diagonal block and for each couple (k, j) the restrictions of the matrices $A'_j(\lambda_{k,j}) - \lambda_{k,j}I$ and $M_j(\sigma_{k,j}) - \sigma_{k,j}I$ to it are equal.

Define the matrices $(D_j)^{-1}A_jD_j$ by the rule for all (k, j) the matrices $A_j(\lambda_{k,j}) - \lambda_{k,j}I$ and $M_j(\sigma_{k,j}) - \sigma_{k,j}I$ to be equal. The rule implies that the JNFs of the matrices $(B_j)^{-1}M_jB_j$ and $(D_j)^{-1}A_jD_j$, hence, of M_j and A_j , coincide. As there are no non-zero integer differences between eigenvalues of A_j , one has also $\exp(2\pi i A_j) = M_j$ (up to conjugacy).

4⁰. The tuple of matrices A_j thus constructed might fail to be with trivial centralizer. Hence, the tuple must define a direct sum of representations (this follows from A_1 being with distinct eigenvalues). So conjugate it to a block-diagonal form where each block (we call these blocks *big blocks*) is small-block upper-triangular and with trivial centralizer. The small blocks are of sizes s_i .

As in Lemma 24 from [Ko3] one shows that if there are two big blocks of sizes u, v where $u \geq 3, v \geq 2$, then one can deform the tuple into one in which these two big blocks are replaced by a single big block of size $u + v$ (with trivial centralizer and with the same small blocks as the two big blocks) while the other big blocks remain the same. The statement holds also if $u = v = 2, p = 2$ (see again Lemma 24 from [Ko3]) and for at least one index $j \geq 2$ the restrictions of the tuple to the two big blocks belong to different conjugacy classes, or if $u = v = 2, p \geq 3$ and no matrix is scalar.

If there is a big block B of size 1, then it follows from $r_2 + \dots + r_{p+1} \geq n + 1$ that for at least one of the other big blocks B' one has $\text{Ext}^1(B, B') \geq 1$. Indeed, without loss of generality one can assume that the restrictions of the matrices to the block B equal 0 for all values of j . Hence, for each other big block B' one has $\text{Ext}^1(B, B') = \rho(B') - 2\sigma(B')$ where $\rho(B')$ is the sum of the ranks $r_j(B')$ of the matrices $A_j|_{B'}$ and $\sigma(B')$ is the size of B' . (One subtracts $\sigma(B')$ once because the sum of the matrices A_j is 0 and once to factor out conjugation with block upper-triangular matrices; see the proof of Lemma 28.)

If for all blocks B' one has $\text{Ext}^1(B, B') \leq 0$, then one has

$$0 \geq \sum_{B'} (\rho(B') - 2\sigma(B')) = \left(\sum_{j=1}^{p+1} \sum_{B'} r_j(B') \right) - 2(n-1) \geq \left(\sum_{j=1}^{p+1} r_j \right) - 2(n-1)$$

i.e. $\sum_{j=2}^{p+1} r_j \leq n - 1$ (recall that $r_1 = n - 1$) which is a contradiction.

Hence, one can replace the two blocks B, B' by a single big block of size $\sigma(B') + 1$.

There remains to be considered the case when there is no big block of size 1 or ≥ 3 , i.e. all big blocks are of size 2; moreover, $p = 2$, and for $j > 1$ the restrictions of the matrices A_j to the big blocks belong to one and the same conjugacy class. In this case one has $r_2 + r_3 = n$, i.e. the case has not to be considered. \square

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